Does the Market Understand Time Variation in the Equity Premium?

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Background

Well-studied set of questions:

▸ What is the expected excess return on the market?
▸ How does it evolve over time?
▸ Are there systematic errors in return predictions?
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- What is the expected excess return on the market?
- How does it evolve over time?
- Are there systematic errors in return predictions?

\[ p_t - d_t = \kappa - \sum_{j=0}^{\infty} \rho_j \mathbb{E}_t r_{t+j+1} + \sum_{j=0}^{\infty} \rho_j \mathbb{E}_t \Delta d_{t+j+1} \]

\[ \mathbb{E}_t r_{t+1} + \sum_{j=1}^{\infty} \rho_j \mathbb{E}_t r_{t+j+1} \]

much more important for pricing!
Background

Well-studied set of questions:

- What is the expected excess return on the market?
- How does it evolve over time?
- Are there systematic errors in return predictions?

\[ p_t - d_t = \kappa - \sum_{j=0}^{\infty} \rho^j \mathbb{E}_t r_{t+j+1} + \sum_{j=0}^{\infty} \rho^j \mathbb{E}_t \Delta d_{t+j+1} \]

Our focus:

- What is the expected future equity premium?
- How does it compare to the actual future equity premium \( \mathbb{E}_{t+j} r_{t+j+1} \)?
- Are there systematic errors in expected return predictions?
What We Do

1. Option-based measure of log equity premium:

   **Spot rate:** \( \mu_t^n = \mathbb{E}_t[r_{t,t+n}] - r_{t,t+n}^f \)

2. Calculate expected future equity premium:

   **Forward rate:** \( f_t^n = \mu_t^{(n+1)} - \mu_t^{(n)} = \mathbb{E}_t[\mu_{t+n}^{(1)}] \)

3. Compare forward rate to realized future spot rate:

   **Forecast error:** \( \varepsilon_{t+n} = \mu_{t+n}^{(1)} - f_t^n \)
Why This Framework?

Spot rate:  \[ \mu_t^{(n)} = \mathbb{E}_t[r_{t,t+n}] - r_{t,t+n}^{f} \]

Forward rate:  \[ f_t^{(n)} = \mu_t^{(n+1)} - \mu_t^{(n)} = \mathbb{E}_t[\mu_{t+n}^{(1)}] \]

Forecast error:  \[ \varepsilon_{t+n} = \mu_{t+n}^{(1)} - f_t^{(n)} \]

- Term structure of log equity premia \( \Rightarrow \) straightforward calculation of forward rates
- Behavior of term structure:
  - Key for prices
  - Great lab for testing whether market-based expectations are intertemporally consistent, \textit{without} needing to take a stand on whether expected returns are themselves rational
- Forecast errors require much weaker conditions than needed to estimate \( \mu_t^{(n)} \) and \( f_t^{(n,m)} \) separately
  - In general, spot and forward rates are contaminated by SDF-related covariances…
  - …but when differencing, the covariance terms largely cancel
What We Find

**Spot rate:** \[ \mu_t^{(n)} = E_t[r_{t,t+n}] - r_{t,t+n}^f \]

**Forward rate:** \[ f_t^{(n)} = \mu_t^{(n+1)} - \mu_t^{(n)} = E_t[\mu_t^{(1)}] \]

**Forecast error:** \[ \epsilon_{t+n} = \mu_{t+n}^{(1)} - f_t^{(n)} \]

In global options sample:

1. Forward rates are strong predictors of realized spot rates
   - Forward rate \( \uparrow \) by 1% \( \implies \) future spot rate \( \uparrow \) by 0.7%
   - Forward rates explain 20% of the variation in future spot rates at 6-month horizon

2. Forecast errors are close to 0 on average...

3. ...but still exhibit predictable mean reversion and excess volatility
   - Can therefore reject fully rational expectations for range of possible SDF specifications
   - Alternatively, need highly volatile and countercyclical price of discount-rate risk
Illustration: U.S. Forward and Realized Spot Rates in Three Crises

1998 Russian Debt Crisis

2008 Financial Crisis

2020 COVID-19 Recession

- **Forward rate at crisis onset**
- **Realized one-month spot rate**
Roadmap

1. Introduction

2. Theory
   - The Log Utility Case
   - The General Case

3. Implementation and Results

4. Rationalizing Forecast Errors

5. A Model of Expectation Errors

6. Discussion and Conclusions
The Log Utility Case: Setup

Setting:

- Representative agent (“the market”)
  - Equivalently, any unconstrained investor who chooses to fully invest in the market
- For now: **Log utility** over wealth

\[
\log(W_{t+n}) = \log(W_t R_{t,t+n})
\]

\[
P_t = \mathbb{E}_t [M_{t,t+n} \text{Payoff}_{t+n}]
\]

\[
M_{t,t+n} = 1/R_{t,t+n}
\]

- Useful for exposition, but will turn out not to be central
The Log Utility Case: Setup

Setting:

- Representative agent (“the market”)
  - Equivalently, any unconstrained investor who chooses to fully invest in the market
- For now: Log utility over wealth $\iff M_{t,t+n} = 1/R_{t,t+n}$
- Building block: $\text{LVIX } l_t^{(n)}$ [Gao and Martin (2021)]:

$$
\mathbb{E}_t[r_{t,t+n}] - r_{t,t+n}^f = (R_{t,t+n}^f)^{-1}\mathbb{E}_t^*[R_{t,t+n}r_{t,t+n}] - r_{t,t+n}^f - \text{Cov}_t(M_{t,t+n}R_{t,t+n}, r_{t,t+n})
$$

- $l_t^{(n)}$: Observable from options (details later)
- Second term: $MR = 1 \implies \text{Cov} = 0$ under log utility
The Log Utility Case: Identification

General notation:

Spot rate: \( \mu_t^{(n)} = \mathbb{E}_t[r_{t,t+n}] - r_{t,t+n} \)
Forward rate: \( f_t^{(n,m)} = \mu_t^{(n+m)} - \mu_t^{(n)} = \mathbb{E}_t[\mu_{t+n}^{(m)}] \)
Forecast error: \( \varepsilon_{t+n}^{(m)} = \mu_{t+n} - f_t^{(n,m)} \)

Timeline:

\( \mu_t^{(n+m)} \)
\( \mu_t^{(n)} \)
\( f_t^{(n,m)} \)
\( t \quad t+n \quad t+n+m \)
The Log Utility Case: Identification

General notation:

Spot rate: \( \mu_t^{(n)} = \mathbb{E}_t[r_{t,t+n}] - r_{t,t+n}^f \)
Forward rate: \( f_t^{(n,m)} = \mu_t^{(n+m)} - \mu_t^{(n)} = \mathbb{E}_t[\mu_t^{(m)}] \)
Forecast error: \( \epsilon_{t+n}^{(m)} = \mu_t^{(m)} - f_t^{(n,m)} \)

Result 1

Given \( M_{t,t+n} = 1/R_{t,t+n} \):

\[
\begin{align*}
\mu_t^{(n)} &= \mathcal{L}_t^{(n)} \\
f_t^{(n,m)} &= \mathcal{L}_t^{(n+m)} - \mathcal{L}_t^{(n)} \\
\epsilon_{t+n}^{(m)} &= \mathcal{L}_{t+n}^{(m)} - \mathcal{L}_{t}^{(n+m)} + \mathcal{L}_t^{(n)}
\end{align*}
\]

- Can then examine term structure, test \( \mathbb{E}[\epsilon_{t+n}^{(m)}] = 0, \mathbb{E}[Z_t \epsilon_{t+n}^{(m)}] = 0, \ldots \)
The General Case: Identification Challenge

Beyond log utility?

- In general, spot and forward rates are contaminated by unobservable covariances:

\[
\mu_t^{(n)} = \mathcal{L}_t^{(n)} - \text{Cov}_t(M_{t,t+n}R_{t,t+n}, r_{t,t+n})
\]

\[
f_t^{(n,m)} = \mathcal{L}_t^{(n+m)} - \mathcal{L}_t^{(n)} + C_t^{(n)} - C_t^{(n+m)}
\]

- For \(\mu_t^{(n)}\), can argue \(C_t^{(n)} \leq 0\) [Gao & Martin (2021)]. . .but for \(f_t^{(n,m)}\), \(C_t^{(n+m)} \gg C_t^{(n)}\)

- Key insight: Covariance terms largely cancel when considering forecast errors \(\epsilon_{t+n}^{(m)} = \mu_{t+n} - f_t^{(n,m)}\).

E.g., for \(n = m = 1\):

\[
\mathbb{E}_t[C_{t+1}^{(1)}] + C_t^{(1)} - C_t^{(2)} = \text{Cov}_t(M_{t,t+1}R_{t,t+1}, \mathbb{E}_{t+1}[r_{t+1,t+2}])
\]

much less volatile than \(r_{t+1,t+2}\).
The General Case: Result

Define forecast-error proxy from log utility case:

\[ \hat{\epsilon}_{t+n}^{(m)} = \mathcal{L}_{t+n}^{(m)} - \mathcal{L}_{t+n}^{(n+m)} + \mathcal{L}_{t}^{(n)} \]

Result 2

For any SDF \( M_{t,t+n} \),

\[ \mathbb{E}_t[\hat{\epsilon}_{t+n}^{(m)}] = \mathbb{E}_t[\epsilon_{t+n}^{(m)}] - \text{Cov}_t(M_{t,t+n}R_{t,t+n}, \mu_{t+n}^{(m)}) \]

- Covariance term now relates to pricing of discount-rate risk, rather than realized-return risk
- Basic idea of proof: \( MR_{t,t+n} \) is orthogonal to unexpected component of \( r_{t+n,t+n+m} \), so left with expectation term \( \mu_{t+n}^{(m)} \)
- Remaining covariance is likely quite small, but can be disciplined empirically or theoretically
The General Case: Result

Define forecast-error proxy from log utility case:

\[
\tilde{e}_{t+n}^{(m)} = \mathcal{L}_{t+n}^{(m)} - \mathcal{L}_{t}^{(n+m)} + \mathcal{L}_{t}^{(n)}
\]

Result 2

For any SDF \( M_{t,t+n} \),

\[
\mathbb{E}_t[\tilde{e}_{t+n}^{(m)}] = \mathbb{E}_t[\varepsilon^{(m)}_{t+n}] - \text{Cov}_t \left( M_{t,t+n} R_{t,t+n}, \mu_{t+n}^{(m)} \right)
\]

Further, define forecast-revision proxy (change in forward rates):

\[
\tilde{\eta}_{t+1}^{(m)} = \left( \mathcal{L}_{t+1}^{(n+m-1)} - \mathcal{L}_{t}^{(n+m)} \right) - \left( \mathcal{L}_{t+1}^{(n-1)} - \mathcal{L}_{t}^{(n)} \right)
\]

Result 3

For any SDF \( M_{t,t+1} \),

\[
\mathbb{E}_t[\tilde{\eta}_{t+1}^{(m)}] = \mathbb{E}_t[\eta^{(m)}_{t+1}] - \text{Cov}_t \left( M_{t,t+1} R_{t,t+1}, f_{t+1}^{(n+m-1)} \right)
\]
Roadmap

1. Introduction

2. Theory

3. Implementation and Results
   - Data
   - Main Estimates
   - Forecast Errors and Predictability
   - Interpretation

4. Rationalizing Forecast Errors

5. A Model of Expectation Errors

6. Discussion and Conclusions
Data and Measurement

Data:
- Global panel of index options from OptionMetrics
  - For U.S. sample: 1990–2021
  - For international sample: Consider 10 major indices, with data since at least 2006
- Sample monthly and apply standard filters
- Baseline: 6-month horizon, 6 months forward \((n = m = 6)\)

Measuring LVIX: Following Gao & Martin (2021), Carr & Madan (2001),

\[
\mathcal{L}^{(n)}_t = \left( R^{f}_{t,t+n} \right)^{-1} \mathbb{E}^*_t \left[ R_{t,t+n} r_{t,t+n} \right] - r^{f}_{t,t+n}
\]

\[
= \frac{1}{P_t} \left\{ \int_{0}^{F^{(n)}_t} \frac{\text{put}^{(n)}(K)}{K} dK + \int_{F^{(n)}_t}^{\infty} \frac{\text{call}^{(n)}(K)}{K} dK \right\}
\]

- Calculate integral a bunch of different ways (appendix has details)
- First: Simplify by working under log assumption, so LVIX \(\implies\) spot & forward rates
Estimates: Contemporaneous U.S. Spot and Forward Rates

Annualized Percent

Current Spot Rate

Forward Rate

1990  2000  2010  2020
Estimates: Realized U.S. Spot and Forward Rates

![Graph showing realized spot rates and forward rates over time with shaded areas indicating forecast errors.](image-url)
Do Forward Rates Predict Future Spot Rates?

Mincer–Zarnowitz Regressions for Spot Rates by Country

\[
\mu_{t+6}^{(6)} = \beta_0 + \beta_1 f_t^{(6,6)} + \epsilon_{t+6}
\]

<table>
<thead>
<tr>
<th></th>
<th>(1) U.S.</th>
<th>(2) Ex-U.S.</th>
<th>(3) All</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_t^{(6,6)} )</td>
<td>0.67***</td>
<td>0.55***</td>
<td>0.56***</td>
</tr>
<tr>
<td></td>
<td>(0.096)</td>
<td>(0.056)</td>
<td>(0.055)</td>
</tr>
<tr>
<td>Intercept</td>
<td>0.74***</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>(0.28)</td>
<td></td>
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<tr>
<td>Country FEs</td>
<td>✗</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>p-value: ( \beta_1 = 1 )</td>
<td>0.003</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Obs.</td>
<td>378</td>
<td>1,849</td>
<td>2,227</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.22</td>
<td>0.21</td>
<td>0.22</td>
</tr>
<tr>
<td>Within ( R^2 )</td>
<td>—</td>
<td>0.14</td>
<td>0.15</td>
</tr>
</tbody>
</table>

SEs in (1) are heteroskedasticity and autocorrelation-robust [Lazarus et al. (2018)], and in (2)–(3) are clustered by exchange and month.

- Substantial predictive power...
- ...but \( \beta_1 \neq 1 \), suggesting forward rates overshoot future spot rates
- What if \( \beta_1 \) estimate is downwardly biased due to measurement error?
- To address this, now consider IV using shorter-term forward rate \( f_t^{(2,1)} \) as instrument for \( f_t^{(6,6)} \)
- Shorter-horizon forwards likely to be better measured: denser option strikes & more trading volume
Do Forward Rates Predict Future Spot Rates?

Instrumented Mincer–Zarnowitz Regressions for Spot Rates

\[ \mu_{t+6}^{(6)} = \beta_0 + \beta_1 f_t^{(6,6)} + \epsilon_{t+6}, \quad f_t^{(6,6)} = \pi_0 + \pi_1 f_t^{(2,1)} + \eta_t \]

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<td>All</td>
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<tr>
<td>( f_t^{(6,6)} )</td>
<td>0.73***</td>
<td>0.69***</td>
<td>0.70***</td>
</tr>
<tr>
<td></td>
<td>(0.062)</td>
<td>(0.078)</td>
<td>(0.074)</td>
</tr>
<tr>
<td>Intercept</td>
<td>0.59***</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(0.13)</td>
<td>-</td>
<td>-</td>
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<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>( p )-value: ( \beta_1 = 1 )</td>
<td>0.018</td>
<td>0.004</td>
<td>0.003</td>
</tr>
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SEs in (1) are heteroskedasticity and autocorrelation-robust [Lazarus et al. (2018)], and in (2)–(3) are clustered by exchange and month.

- Forward rate ↑ by 1%

\[ \Rightarrow \text{future spot rate ↑ by } \sim 0.7\% \]

- Forward rates explain \sim 20% of the variation in future spot rates

- Now soften log utility assumption and turn to forecast errors
Average Forecast Errors Are Close to Zero

\[ \varepsilon_{t+6}^{(6)} = \mu_{t+6}^{(6)} - f_t^{(6,6)} \]

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<td>Average</td>
<td>0.021 (0.15)</td>
<td>0.20 (0.11)</td>
<td>0.17 (0.11)</td>
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SEs in (1) are HAR [Lazarus et al. (2018)], and in (2)–(3) are clustered by exchange and month.

- Not just statistically insignificant, but effectively zero: \( \bar{\varepsilon} \leq 20 \) bps annualized
- Therefore can’t reject log utility + RE just on the basis of average errors
  - Not the highest-powered test, but will be informative in trying to rationalize time variation
- But average of zero masks substantial predictability…
Forecast Errors and Lagged Forward Rates Over Time

Annualized Percent

Forecast Error

Lagged Forward Rate (Demeaned)
Forward Rates as Predictors of Forecast Errors

**FORWARD RATES (IV)**

**PREDICTED FORECAST ERRORS**

Annualized Percent
Predictable Forecast Errors

Regressions of Forecast Errors on $2 \times 1$ Forward Rate

\[ \varepsilon_{t+6}^{(6)} = \beta_0 + \beta_1 f_t^{(2,1)} + e_{t+6} \]

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Predictable Forecast Errors

### Regressions of Forecast Errors on 2×1 Forward Rate

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\varepsilon_{t+6}^{(6)} = \beta_0 + \beta_1 f_t^{(2,1)} + \varepsilon_{t+6}
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*SEs in (1) are HAR [Lazarus et al. (2018)], and in (2)–(3) are clustered by exchange and month.*

- Forward rates again overshoot future spot rates
- Errors are also predictable in Coibion–Gorodnichenko regressions using forward-rate revisions (more shortly)
- And predictability rises substantially \((R^2 = 0.11)\) with kernel regression: Arises mostly from high forward rates
- Is this consistent with “overreaction”? It depends: Overreaction to what?
- Option-based expected returns: Yes \([\text{Spot rates, fwd rates, fwd-rate revisions}]\)
- Past returns: Wrong direction!
Reminder: Forecast Errors and Lagged Forward Rates

- Russian Financial Crisis
- Stock Market Downturn of 2002
- Global Financial Crisis
- Double-Dip Recession Fears
- Debt-Ceiling Crisis
- Covid-19 Recession

Annualized Percent

1990 2000 2010 2020

Forecast Error
Lagged Forward Rate (Demeaned)
How Significant Are Forecast Errors?

Before taking any stand on source of estimated forecast errors [*expectation errors vs. risk premia*], return to question posed at outset: **How significant are they for price variation?**

\[
pt - dt = \kappa - \sum_{j=0}^{\infty} \rho^j \mathbb{E}_t r_{t+j+1} + \sum_{j=0}^{\infty} \rho^j \mathbb{E}_t \Delta d_{t+j+1}
\]
How Significant Are Forecast Errors?

Before taking any stand on source of estimated forecast errors [expectation errors vs. risk premia], return to question posed at outset: **How significant are they for price variation?**

\[ p_t - d_t = \kappa - \mathbb{E}_t r_{t+1} - \sum_{j=1}^{\infty} \rho^j f_t^{(j,1)} - \sum_{j=0}^{\infty} \rho^j \mathbb{E}_t \Delta d_{t+j+1} \]

Break \( f_t^{(j,1)} \) into:

\( f_t^{(j,1)} = \mathbb{E}_t[\mu_{t+j}^{(1)}] + \mathbb{E}_t[\varepsilon_{t+j}^{(1)}] \)

\( \Rightarrow \) expected spot rates + predictable forecast errors

- Set one period to be 3 months, and predict \( f_t^{(1)} \) for \( j = 2, 3 \) (6 & 9 months ahead) using 2m×1m forward

- Assume errors follow \( \mathbb{E}_t[\varepsilon_{t+j+1}^{(1)}] = \phi^j \mathbb{E}_t[\varepsilon_{t+j}^{(1)}] \) [De la O & Myers (2021)] \( \Rightarrow \) \( \phi \approx 1 \)

- Use this to calculate discounted sum of predicted forecast errors’ maximal possible effect on prices

- Compare to repurchase-adj. \( p_t - d_t \) from Nagel & Xu (2022)
Discounted Forecast Errors and Price-Dividend Variation

Meaningful in magnitude, esp. during crisis, and overall accounts for 8% of $p_t - d_t$ variation
Roadmap

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3. Implementation and Results

4. Rationalizing Forecast Errors

5. A Model of Expectation Errors

6. Discussion and Conclusions
Can Forecast Errors Be Rationalized?

\[
\mathbb{E}_t[\varepsilon_{t+n}^{(m)}] = \mathbb{E}_t[\varepsilon_{t+n}^{(m)}] - \text{Cov}_t(M_{t,t+n}R_{t,t+n}, \mu_{t+n}^{(m)})
\]

What conditions do we need on \( \zeta_t \) in order for expectation errors \( \mathbb{E}_t[\varepsilon_{t+n}^{(m)}] \) to be unpredictable? Must have \( -\zeta_t \) take same sign as pred. forecast errors:

Main challenge: Small on average, but must flip signs dramatically (− in good times, + in bad).
Can Forecast Errors Be Rationalized?

$$\mathbb{E}_t[\hat{e}_{t+n}^{(m)}] = \mathbb{E}_t[e_{t+n}^{(m)}] - \text{Cov}_t\left(M_{t,t+n}R_{t,t+n}, \mu_{t+n}^{(m)}\right)$$

What conditions do we need on $\zeta_t$ in order for **expectation errors** $\mathbb{E}_t[e_{t+n}^{(m)}]$ to be unpredictable?

- For simplicity: Take $n = m = 1$, and assume $M_{t+1}, R_{t+1}, \mu_{t+1}$ jointly log-normal
- Then $\zeta_t > 0$ (as needed in bad times) if and only if:

$$SR_t(-\mu_{t+1}) > -\rho_t(r, \mu)\sigma_t(r),$$

where $SR_t$ is Sharpe ratio on claim to next period’s negative equity premium (low payoff is bad)

- Correlation $\rho_t(r, \mu)$ likely to be negative; for illustration, set it to $-1$
- Then $SR_t$ must vary *more than* $\sigma_t(r)$ for $\zeta_t$ to flip signs
- One calibration: Go back to log utility (likely to be conservative for time variation in $\sigma_t$), and estimate $\sigma_t(r)$ from options
Can Forecast Errors Be Rationalized?

$\mathbb{E}_t[\varepsilon_{t+n}^{(m)}] = \mathbb{E}_t[\varepsilon_{t+n}] - \text{Cov}_t(M_{t,t+n}R_{t,t+n}, \mu_{t+n}^{(m)})$

What conditions do we need on $\zeta_t$ in order for expectation errors $\mathbb{E}_t[\varepsilon_{t+n}^{(m)}]$ to be unpredictable?

- $SR_t$ must vary more than $\sigma_t(r)$ for $\zeta_t$ to flip signs
- One calibration: Go back to conservative log utility case, and estimate $\sigma_t(r)$ from options. Results for conditional volatility of 6-month market return:
Can Forecast Errors Be Rationalized?

\[
\mathbb{E}_t[\hat{\epsilon}_{t+n}^{(m)}] = \mathbb{E}_t[\epsilon_{t+n}^{(m)}] - \text{Cov}_t\left(M_{t,t+n}R_{t,t+n}, \mu_{t+n}^{(m)} \right) \\
\xi_t
\]

What conditions do we need on \( \xi_t \) in order for expectation errors \( \mathbb{E}_t[\epsilon_{t+n}^{(m)}] \) to be unpredictable?

- \( SR_t \) must vary more than \( \sigma_t(r) \) for \( \xi_t \) to flip signs
- One calibration: Go back to conservative log utility case, and estimate \( \sigma_t(r) \) from options
- More generally, difficult to get both average errors (small) and time variation (large) right
- Paper has one illustration varying risk aversion \( \gamma \)
Roadmap

1. Introduction
2. Theory
3. Implementation and Results
4. Rationalizing Forecast Errors
5. A Model of Expectation Errors
6. Discussion and Conclusions
Model Setup

▶ Now want a simple lab to examine whether findings could plausibly arise from combo of:

1. Log utility
2. Expectation errors

⇒ consider a version of framework in Bordalo, Gennaioli, Shleifer (2018), Augenblick & Rabin (2021)

▶ 3-month spot rate dynamics under objective measure:

\[
\mu_t^{(3)} = \left(1 - \sum_{j=1}^{3} \phi_j \right) \bar{\mu} + \phi_1 \mu_{t-1}^{(3)} + \phi_2 \mu_{t-2}^{(3)} + \phi_3 \mu_{t-3}^{(3)} + \epsilon_t, \quad \epsilon_t \text{i.i.d.} \sim \mathcal{N}(0, \sigma^2_{\epsilon})
\]

▶ Under RE: Term structure of current spot rates would be based on objective expectations \( \mathbb{E}_t \left[ \mu_{t+n}^{(3)} \right] \)

▶ Actual subjective expectations: Excess sensitivity to news governed by “diagnosticity” parameter \( \theta \):

\[
\mathbb{E}_t^{\theta} \left[ \mu_{t+n}^{(3)} \right] = \mathbb{E}_t \left[ \mu_{t+n}^{(3)} \right] + \theta \left( \mathbb{E}_t \left[ \mu_{t+n}^{(3)} \right] - \mathbb{E}_{t-3} \left[ \mu_{t+n}^{(3)} \right] \right)_{\text{news } \propto \epsilon_t}
\]
Model Setup

\[
\mu_t^{(3)} = \left(1 - \sum_{j=1}^{3} \phi_j\right) \bar{\mu} + \phi_1 \mu_{t-1}^{(3)} + \phi_2 \mu_{t-2}^{(3)} + \phi_3 \mu_{t-3}^{(3)} + \epsilon_t, \quad \epsilon_t \sim \text{i.i.d. } \mathcal{N}(0, \sigma_{\epsilon}^2)
\]

\[
\mathbb{E}_t^{\theta} \left[ \mu_{t+n}^{(3)} \right] = \mathbb{E}_t \left[ \mu_{t+n}^{(3)} \right] + \theta \left( \mathbb{E}_t \left[ \mu_{t+n}^{(3)} \right] - \mathbb{E}_{t-3} \left[ \mu_{t+n}^{(3)} \right] \right)
\]

- Forward rates based on subjective expectations
- Longer-term spot rates embed objective short rate and subjective expectations of future short rates
- Consider a range of values for \(\theta\)
  - \(\theta = 0\): RE
  - \(\theta = 0.91\): BGS (2018)
- Estimate objective parameters for spot-rate process in each country
- For each \(\theta\), simulate 10,000 samples and run same tests as in the data for \(n, m = 6\) months
Model vs. Data: Main Estimates

Mincer–Zarnowitz Regressions

Predictability of Forecast Errors

Average Forecast Errors

Coibion–Gorodnichenko Regressions

Slope $\beta_1$

Sensitivity Parameter $\theta$

Slope $\beta_1$

Sensitivity Parameter $\theta$

Model & 95% & Data

RE: $\theta = 0$ & BGS: $\theta = 0.91$
Model vs. Data: $R^2$ Values

Simple calibration does reasonably well on main estimates…
…but seems to miss some rational variation in forward rates:

Legend

- Model
- 95% Confidence Band
- Data
- RE: $\theta = 0$
- BGS: $\theta = 0.91$
A Trilemma for Expectation Errors

More generally:

- While simple calibrated model does reasonably well at matching the data, again not an unqualified success for all possible notions of overreaction

- Subjective beliefs overreact to increases in *spot rates* in our model, not past returns, and cyclicity matters:

  \[ p_t - d_t = \kappa - \sum_{j=0}^{\infty} \rho^j E_t r_{t+j+1} \| E_t r_{t+1} - \sum_{j=1}^{\infty} \rho^j f_t^{(j,1)} + \sum_{j=0}^{\infty} \rho^j E_t \Delta d_{t+j+1} - RF_t \]

- Use \( \sim \) to denote *expectation error wedge* (deviation from RE economy):

  \[ \text{var}(\sim p_t - \sim d_t) = \text{var}(\sim F_t) + \text{var}(\sim CF_t) - 2 \text{cov}(\sim F_t, \sim CF_t) \]

- Have to choose between **two of three**:
  1. Volatile expectation errors for cash flows and/or returns
  2. Volatile price-dividend ratio relative to RE
  3. Positive comovement between fundamental and return expectation errors
Roadmap

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Final Notes

Summary:
- Introduce new methodology to test whether the market understands time variation in equity premium
- Find evidence that it does...to an extent

Tie-ins:
- Equity and fixed-income term structure
- Our tests are similar to tests of the expectations hypothesis, but with less room for discount-rate variation than in previous versions
- Similar to past work [van Binsbergen & Koijen (2017), Gormsen (2021)], find more predictability in equity term structure than in FI term structure
- Also build on Giglio & Kelly (2018) work on other term structures

Still to do: Additional tests, more work on potential rational discount-rate variation, ...