Current Topics in Finance

Week 1: Semiparametric Asset-Pricing Tests

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Background

Oversimplification of *(some moderately sized subset of)* **the asset-pricing literature:**

1. Write down fully specified model of the world
2. Plug some numbers in
3. See if you can replicate data, *especially* some puzzling aspect(s) of data

What's the goal here?

- Use asset prices as relevant info about the underlying structure of the world
- If your model explains the data, then maybe we've learned something about beliefs, preferences, stochastic processes we care about, . . .

My approach instead has been:

- Make some general(ish) assumptions that allow for statements of the form:

  $$\text{data} \implies \text{underlying structure of the world}$$

  without fully specified model
- This is not my invention! Lots of extremely well-known papers take this form

Background

Review of one example: Hansen & Jagannathan (1991)

[simpler version following Shiller (1982), Hansen (1982)]

- Consider a single risky asset (i) and risk-free asset (f)
- Pricing equation and Cauchy-Schwartz inequality give:

$$\begin{aligned} & \mathbb{E}[M(R_i - R_f)] = 0 \\ \iff \quad & \operatorname{Cov}(M, R_i - R_f) + \mathbb{E}[M]\mathbb{E}[R_i - R_f] = 0 \\ \iff \quad & \mathbb{E}[R_i - R_f] = \frac{-\operatorname{Cov}(M, R_i - R_f)}{\mathbb{E}[M]} = \frac{-\operatorname{Corr}(M, R_i - R_f)\sigma(M)\sigma(R_i - R_f)}{\mathbb{E}[M]} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \leqslant \frac{\sigma(M)\sigma(R_i - R_f)}{\mathbb{E}[M]}, \end{aligned}$$

so $\operatorname{SR}(R_i - R_f) \equiv \dfrac{\mathbb{E}[R_i - R_f]}{\sigma(R_i - R_f)} \leqslant \dfrac{\sigma(M)}{\mathbb{E}[M]}$

- Much more general form of equity premium puzzle: setting $i = \text{market}$, $\mathbb{E}[M] = \frac{1}{R_f}$, get lower bound for $\sigma(M)$ around 0.4 (and can be made sharper)

Background

Question I became interested in:

- How restrictive is the assumption of rational expectations?
  - Can we come up with bounds on primitives under general conditions?
  - Yes, and bounds end up being informative
  - Either (a) RE doesn't hold, (b) condition under which bound is derived is strongly violated, or (c) risk aversion is extremely high
- Bound relates asset-price volatility to risk aversion required to rationalize that volatility
- Why use *volatility* as being informative about expectations?
  1. First became interested after "Taper Tantrum" mid-2013
  2. Long history of volatility bounds, following Shiller (1981)
  3. Encountered a useful starting point (Augenblick & Rabin, 2018) at the right time
- Done with (most of) throat-clearing — remainder of slides go through JMP

Introduction

How restrictive is the assumption of **rational expectations** in asset markets?

- **Joint hypothesis problem** $\Longrightarrow$ no free answers
- Illustration: Volatility bounds [e.g., Shiller (1981)]:

$$P_t + \text{error} = \text{ex-post fundamental value}$$

$$\Longrightarrow \quad \text{Var}(P_t) < \text{Var}(\text{ex-post fundamental value}) \qquad \text{[Theory]}$$

$$\text{Var}(P_t) > \text{Var}\left(\sum_{j=1}^{\infty} \frac{D_{t+j}}{R^j}\right) \qquad \text{[Data]}$$

constant discount rates

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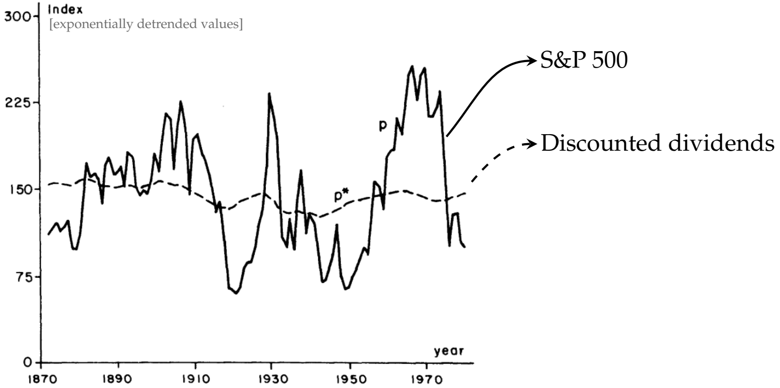

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constant discount rates

- Response [Fama (1991)]:

  "Volatility tests are a useful way to show that expected returns vary, [but] give ***no*** *help on the central issue of whether the variation in expected returns is rational."*

- Can we say anything with weaker assumptions? **Yes.**

Our Contributions

1. **Theory:** In general framework, derive bound under RE:

$$\underbrace{\text{Variation in market-implied beliefs}}_{\text{[risk-neutral beliefs]}} \leqslant f(\underbrace{\text{risk aversion}}_{\text{[SDF slope]}})$$

2. **Evidence:**

Our Contributions

1. **Theory:** In general framework, derive bound under RE:

$$\underset{\text{[second moment]}}{\text{Variation in market-implied beliefs}} \leqslant \underset{\text{[SDF slope]}}{f(\text{risk aversion})}$$

   - N.B. comparison with Hansen–Jagannathan (1991) bound:

$$\underset{\text{[first moment]}}{\text{Sharpe ratio for returns}} \leqslant \underset{\text{[SDF \textit{volatility}]}}{g(\text{risk aversion})}$$

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2. **Evidence:**
   - S&P index options
   - Volatile risk-neutral beliefs $\Longrightarrow$ very high required risk aversion

Possible objections:

(a) Restriction: $\mathbb{E}_t[U'_T \mid \text{return state } a] / \mathbb{E}_t[U'_T \mid \text{return state } b]$ constant over t
   - Risk-neutral belief volatility is still an informative moment

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(a) Restriction: $\mathbb{E}_t[U'_T \mid \text{return state } a] / \mathbb{E}_t[U'_T \mid \text{return state } b]$ constant over t

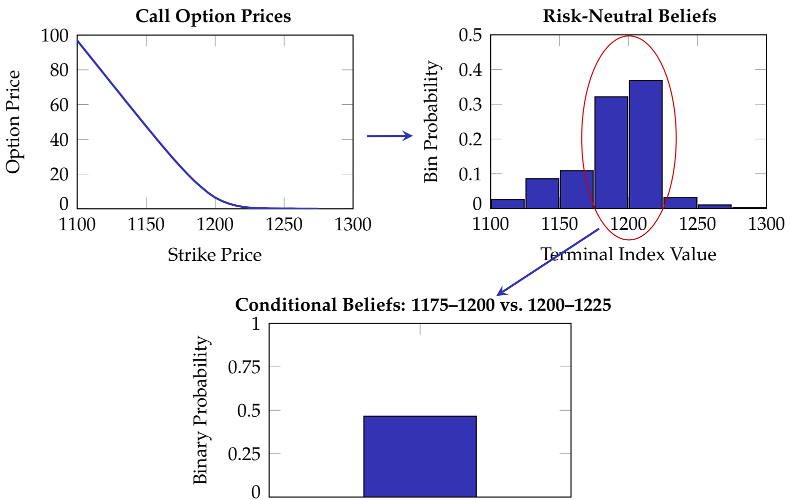
(b) Measurement noise in option prices

- Will address directly

Preview: Empirical Variation

S&P 500 Option Prices and Risk-Neutral Beliefs as of July 1, 2005

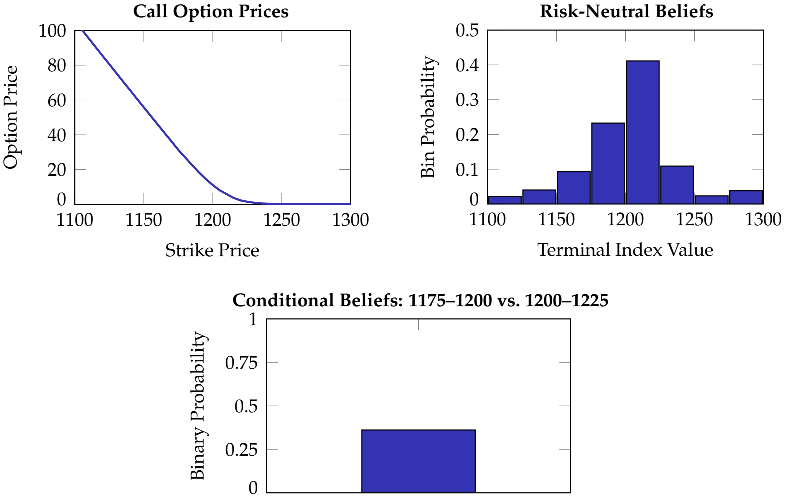
Expiration Date: July 16, 2005



Preview: Empirical Variation

S&P 500 Option Prices and Risk-Neutral Beliefs as of July 5, 2005

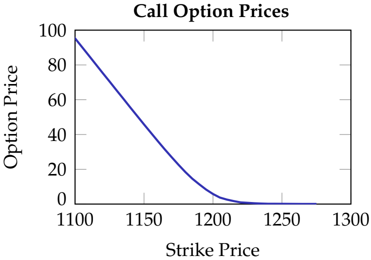
Expiration Date: July 16, 2005

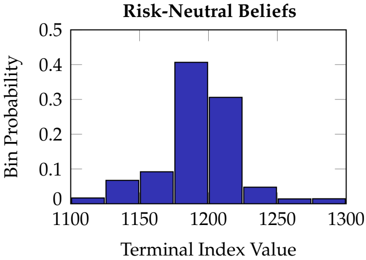


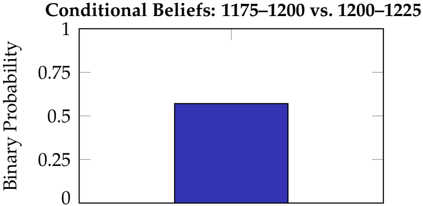
Preview: Empirical Variation

S&P 500 Option Prices and Risk-Neutral Beliefs as of July 6, 2005

Expiration Date: July 16, 2005



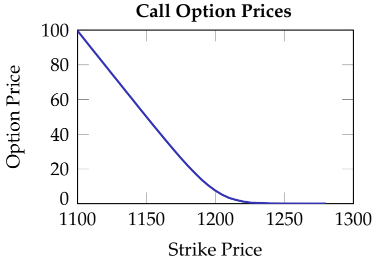


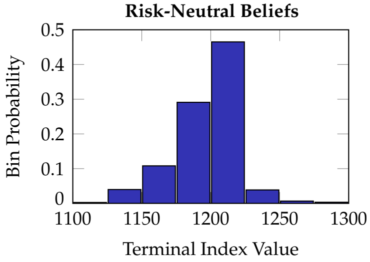


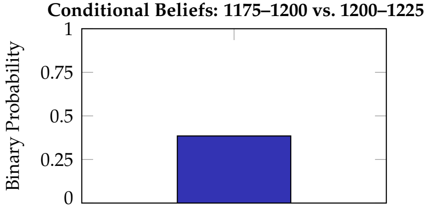
Preview: Empirical Variation

S&P 500 Option Prices and Risk-Neutral Beliefs as of July 7, 2005

Expiration Date: July 16, 2005



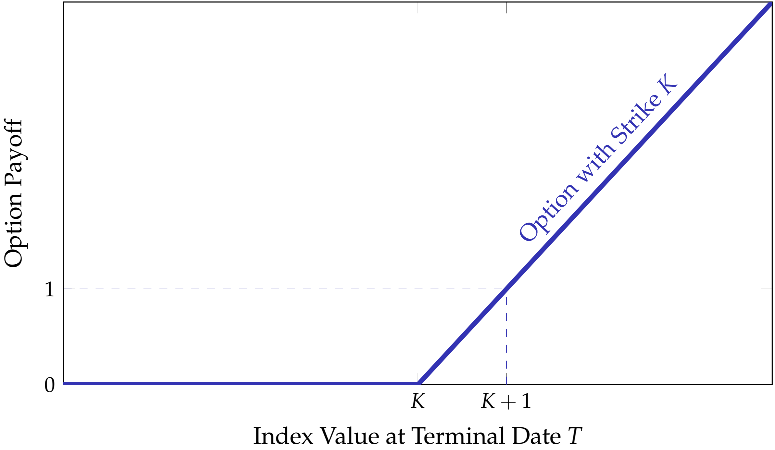




Graphical Intuition: Risk-Neutral Beliefs

Background: Index option prices $\Longrightarrow$ risk-neutral beliefs over future index price

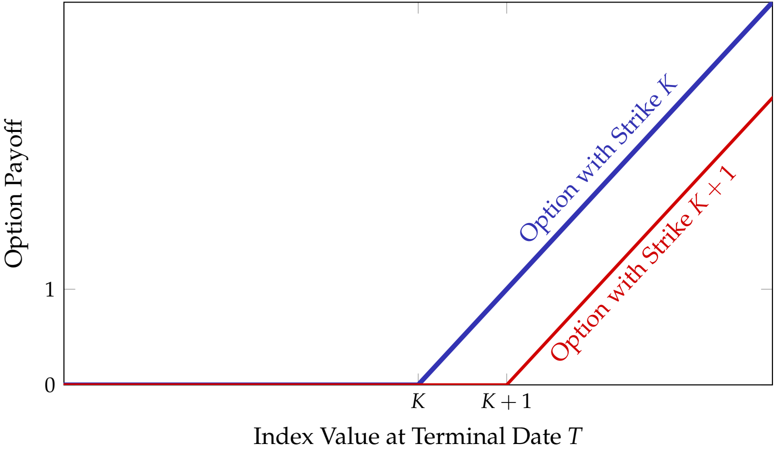
- On date t, can buy or sell **call option** expiring at T with strike K. Payoff:



Graphical Intuition: Risk-Neutral Beliefs

Background: Index option prices $\Longrightarrow$ risk-neutral beliefs over future index price

- On date t, can buy or sell **call option** expiring at T with strike K
- With strike $K+1$?

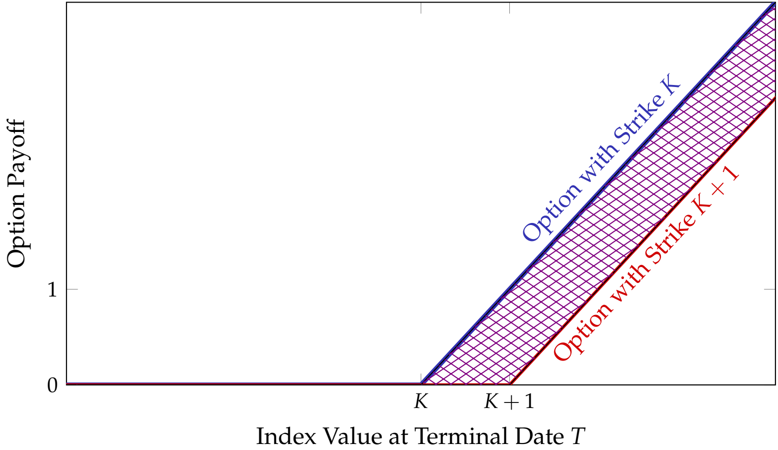


Graphical Intuition: Risk-Neutral Beliefs

Background: Index option prices $\Longrightarrow$ risk-neutral beliefs over future index price

- Payoff to buying option with strike K + selling strike $K+1 \approx \mathbb{1}\{\text{Index}_T \geqslant K\}$

  $\Longrightarrow$ like an Arrow-Debreu security for the "state" $\{\text{Index}_T \geqslant K\}$

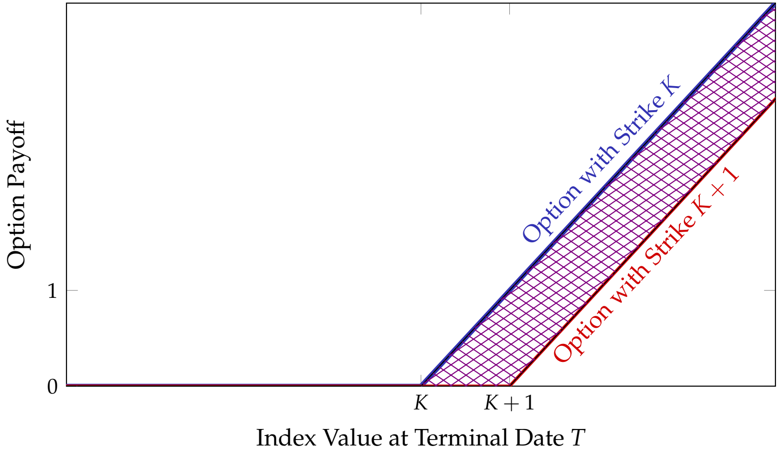


Graphical Intuition: Risk-Neutral Beliefs

Background: Index option prices $\Longrightarrow$ risk-neutral beliefs over future index price

- Payoff to buying option with strike K + selling strike $K+1 \approx \mathbb{1}\{\text{Index}_T \geqslant K\}$

$$\Longrightarrow \text{Price}_t \approx \mathbb{E}_t^*[\mathbb{1}\{\text{Index}_T \geqslant K\}] = \pi_t^*(\text{Index}_T \geqslant K)$$



Outline

1. Background & Intro
2. Theory: Restrictions Under Rational Expectations
   - Two-State Example
   - General Bounds
3. Empirics: Evidence from Index Options
4. Discussion and Conclusions

Two-State Example: Directly Observed Beliefs

Assumptions (*dropped in general framework*):

- Representative agent:
  1. Risk-neutral, no discounting
  2. Random terminal consumption: $C_T = C_{\text{low}}$ or C_{high}
- Beliefs: $\pi_t \equiv$ subjective probability of state C_{low}
- Rational expectations: $\pi_t = \text{Prob}_t(C_T = C_{\text{low}})$
- Arrow-Debreu security with payoff $\mathbb{1}\{C_T = C_{\text{low}}\}$
  - Risk neutrality $\Longrightarrow$ price $= \pi_t$

Two-State Example: Directly Observed Beliefs

Objects we'll keep track of:

1. **Belief movement:** $$m_{t_1,t_2} \equiv \sum_{t=t_1+1}^{t_2} (\pi_t - \pi_{t-1})^2$$
   - "Volatility" $\iff$ sum of squared changes in beliefs
2. **Uncertainty resolution:**
   $$r_{t_1,t_2} \equiv \underbrace{(1-\pi_{t_1})\pi_{t_1}}_{\text{uncertainty}_{t_1}} - \underbrace{(1-\pi_{t_2})\pi_{t_2}}_{\text{uncertainty}_{t_2}}$$
   - "Uncertainty" $\iff$ variance of binomial RV, maximized at 0.5

Restriction (Augenblick & Rabin, 2018):

$$\text{Under RE, } \mathbb{E}[m_{t_1,t_2}] = \mathbb{E}[r_{t_1,t_2}].$$

Derivation

- Formalizes "correct" amount of volatility of subjective beliefs
- Derivation uses only martingale property of beliefs: $\pi_t = \mathbb{E}_t[\pi_{t+1}]$

Two-State Example: Directly Observed Beliefs

Objects we'll keep track of for full path:

1. **Belief movement:** $$m \equiv \sum_{t=1}^{T} (\pi_t - \pi_{t-1})^2$$
   - "Volatility" $\iff$ sum of squared changes in beliefs
2. **Uncertainty resolution:** $$r \equiv \underbrace{(1-\pi_0)\pi_0}_{\text{initial uncertainty}} - \underbrace{(1-\pi_T)\pi_T}_{0}$$
   - "Uncertainty" $\iff$ variance of binomial RV, maximized at 0.5

Restriction (Augenblick & Rabin, 2018):

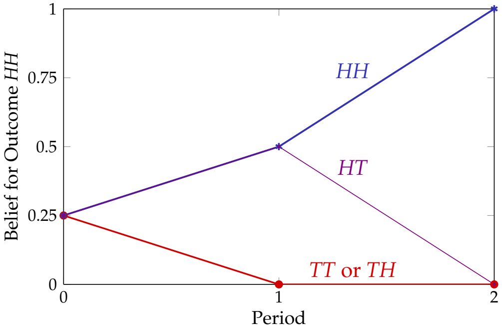
$$\text{Under RE, } \mathbb{E}[m] = r.$$

Derivation

- **Intuition:** Changing beliefs $\iff$ must be learning something (on average)
- Violations can arise from too large (or small) belief revisions

Two-State Example: Directly Observed Beliefs

$$T = 2, \text{ sequential fair coin flips at } t = 1 \text{ and } t = 2,\ C_T = \begin{cases} C_{\text{low}} & \text{if } HH \\ C_{\text{high}} & \text{else} \end{cases}$$



Statistics:

Movement

$$m \equiv \sum_{t=1}^{2} (\pi_t - \pi_{t-1})^2$$

Uncertainty Resolution

$$r \equiv (1 - \pi_0)\pi_0$$

Restriction: $\mathbb{E}[m] = r$

Path	Movement	Frequency
HH	$5/16$	$1/4$
HT	$5/16$	$1/4$
$T*$	$1/16$	$1/2$

$$\Longrightarrow \mathbb{E}[m] = 3/16$$
$$= 3/4 \times 1/4 = r \quad \checkmark$$

Two-State Example with Risk Aversion

Problem: Asset prices don't give us subjective beliefs (joint hypothesis problem).
Assume now:

- Utility: $\mathbb{E}_0 \sum_{t=0}^{T} U(C_t), \;\; U'' < 0$
- Equilibrium: A-D security with payoff $\mathbb{1}\{C_T = C_{\text{low}}\}$ now has price

$$q_t(C_{\text{low}}) = \frac{U'(C_{\text{low}})}{U'(C_t)} \, \pi_t$$

$\Longrightarrow$ subjective beliefs no longer observable

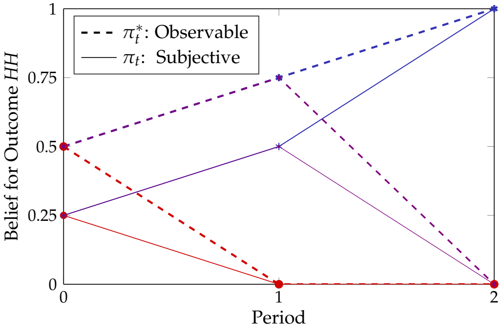
- Define *risk-neutral belief*:

$$\pi_t^* \equiv \frac{q_t(C_{\text{low}})}{q_t(C_{\text{low}}) + q_t(C_{\text{high}})} = \frac{U'(C_{\text{low}})}{\mathbb{E}_t[U'(C_T)]} \, \pi_t \geqslant \pi_t$$

  - Need not follow a martingale under RE $\Longrightarrow$ can have $\mathbb{E}[m^*] > \mathbb{E}[r^*]$

Two-State Example with Risk Aversion

Back to coin-flip example, with $U'(C_{\textbf{low}}) = 3 \times U'(C_{\textbf{high}})$:



Observables:

Movement

$$m^* \equiv \sum_{t=1}^{2} (\pi^*_t - \pi^*_{t-1})^2$$

- Simple variance swap payoff

Uncertainty Resolution

$$r^* \equiv (1 - \pi^*_0)\pi^*_0$$

- Risk-neutral variance

Restriction: $\mathbb{E}[m^* - r^*] \leqslant ??$

Path	m^*	Frequency
HH	~~5/16~~ 1/8	1/4
HT	~~5/16~~ 5/8	1/4
$T*$	~~1/16~~ 1/4	1/2

$$\Longrightarrow \quad \mathbb{E}[m^*] = 5/16$$
$$> (1 - 1/2) \times 1/2$$
$$= r^* \quad ✘$$

Two-State Example: Result

How much excess movement can there be?

Proposition

Under RE,

$$\mathbb{E}_0[m^* - r^*] \leqslant \pi_0^* \underbrace{(\pi_0^* - \pi_0)} = \pi_0^* \left(\pi_0^* - \frac{\pi_0^*}{\pi_0^* + \underbrace{\phi}_{U'(C_{\text{low}})/U'(C_{\text{high}})} (1 - \pi_0^*)} \right)$$

indexes excess movement given downward revisions to beliefs

room for downward movement

General Framework: Setup (I)

Now have to take into account that state space isn't binary.

Probability space:

- Discrete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in \mathbb{N}}$
- Ex-dividend value of market index: $V_t^m \colon \Omega \to \mathbb{R}_+$
- Interested in its value on some option expiration date T
  - Call option with strike K has payoff $\max\{V_T^m - K, 0\}$
- Say that **return state** $s \in \mathcal{S}$ is realized if

$$R_T^m \equiv \frac{V_T^m}{V_0^m} = s$$

- **Objective** probabilities of return states governed by $\mathbb{P} \colon \mathcal{F} \to [0, 1]$

General Framework: Setup (II)

Stochastic discount factor:

- Absence of arbitrage $\Longrightarrow$ existence of strictly positive **stochastic discount factor (SDF)** process $\{M_t\}$ s.t. price S_t of claim to random payoff X_T is

$$S_t(X_T) = \mathbb{E}_t\left[\frac{M_T}{M_t} X_T\right]$$

  - Standard representative-agent economy: $\frac{M_{t+1}}{M_t} = \beta \frac{U'(C_{t+1})}{U'(C_t)}$

- **Risk-neutral measure** for T-dated payoffs: $\left.\frac{d\mathbb{P}^*}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{M_T/M_t}{\mathbb{E}_t[M_T/M_t]}$

Beliefs:

- Assume prices generated by some marginal trader observing public signals
- Agent observes signal vector $\theta_t \in \Theta$, with $\mathcal{F}_t = \sigma(\theta_\tau, 0 \leqslant \tau \leqslant t)$
- **Signal-generating process** $\mathbb{P}(\theta_t \mid \mathcal{F}_{t-1}, R_T^m)$ governs info on return states
- **Belief distribution** over return states: $\{\pi_t(R_T^m = s)\}_{s \in \mathcal{S}}$

General Framework: Setup (III)

Definition

An agent has **rational expectations** over return states at T if and only if both:

(i) Her date-0 priors coincide with the objective probabilities:

$$\pi_0(R_T^m = s) = \mathbb{P}_0(R_T^m = s) \quad \forall s \in \mathcal{S}.$$

(ii) She updates according to **Bayes' rule** using the objective likelihood function:

$$\pi_t(R_T^m = s) = \frac{\pi_{t-1}(R_T^m = s)\, \mathbb{P}(\theta_t | \, \mathcal{F}_{t-1}, R_T^m = s)}{\mathbb{P}(\theta_t | \, \mathcal{F}_{t-1})}.$$

- Under RE, risk-neutral belief distribution:

$$\pi_t^*(R_T^m = s) = \frac{\mathbb{E}_t[M_T/M_t \mid R_T^m = s]}{\mathbb{E}_t[M_T/M_t]}\, \pi_t(R_T^m = s)$$

General Framework: Restriction for Identification

- Work with **conditional** risk-neutral beliefs for state s_j vs. s_{j+1}:

$$\widetilde{\pi}^*_{t,j} \equiv \pi^*_t(R^m_T = s_j \mid R^m_T \in \{s_j, s_{j+1}\}) = \frac{\pi^*_t(R^m_T = s_j)}{\pi^*_t(R^m_T = s_j) + \pi^*_t(R^m_T = s_{j+1})}$$

$$\Longrightarrow \quad \frac{\widetilde{\pi}^*_{t,j}}{1 - \widetilde{\pi}^*_{t,j}} = \underbrace{\frac{\mathbb{E}_t[M_T/M_t \mid R^m_T = s_j]}{\mathbb{E}_t[M_T/M_t \mid R^m_T = s_{j+1}]}}_{:= \phi_{t,j}} \frac{\widetilde{\pi}_{t,j}}{1 - \widetilde{\pi}_{t,j}}$$

Definition

The SDF satisfies **conditional transition independence (CTI)** for the return-state pair (s_j, s_{j+1}) and option expiration date T if $\phi_{t,j}$ is constant for all $0 \leqslant t < T$ almost surely, and we denote this constant by ϕ_j.

- $\widetilde{\pi}^*_{t,j}$ changes must arise from subjective beliefs, not relative severity of states
- Requires that M_T/M_t in s_j depend in expectation only on s_j and not on the path of unobservable state variables between t and T
- $\phi_j \Longleftrightarrow$ utility curvature across states (roughly), and label states so that $\phi_j \geqslant 1$

General Framework: Restriction for Identification

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What models work?

- Variable rare disasters
- Long-run risks
- ~~Habit formation~~

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What models work?

- ~~Habit formation~~

But permanent shocks to the SDF are admissible in all models.

Bound in General Framework & Interpretation

Proposition

$$\widetilde{\mathbb{E}}_0[m_j^* - r_j^*] \leqslant \widetilde{\pi}_{0,j}^{*\,2}\left(1 - \frac{1}{\widetilde{\pi}_{0,j}^* + \underbrace{\phi_j}_{\substack{\cancel{U'(C_{\text{low}})/U'(C_{\text{high}})} \\ \mathbb{E}_t[M_T \,|\, R_T^m = s_j]/\mathbb{E}_t[M_T \,|\, R_T^m = s_{j+1}]}}(1 - \widetilde{\pi}_{0,j}^*)}\right)$$

General framework:

- **Setting:** Uncertainty over terminal value of market index, V_T^m
- **State space:** Many return states $\{s_j\}$ defined by $R_T^m \equiv \frac{V_T^m}{V_0^m} = s_j$
- **Physical beliefs:** $\widetilde{\pi}_{t,j} \equiv \pi_t(R_T^m = s_j \,|\, R_T^m \in \{s_j, s_{j+1}\})$
- **Risk-neutral beliefs:** SDF $\{M_t\} \implies \widetilde{\pi}_{t,j}^* = \dfrac{\mathbb{E}_t[M_T \,|\, R_T^m = s_j]}{\mathbb{E}_t[M_T \,|\, R_T^m \in \{s_j, s_{j+1}\}]}\,\widetilde{\pi}_{t,j}$
- **Identification restriction:** ϕ_j is constant

Robustness & simulations

Bound in General Framework & Interpretation

Proposition

$$\widetilde{\mathbb{E}}_0[m_j^* - r_j^*] \leqslant \widetilde{\pi}_{0,j}^{*\,2}\left(1 - \frac{1}{\widetilde{\pi}_{0,j}^* + \underbrace{\phi_j}_{\substack{\cancel{U'(C_{\text{low}})/U'(C_{\text{high}})} \\ \mathbb{E}_t[M_T \mid R_T^m = s_j]/\mathbb{E}_t[M_T \mid R_T^m = s_{j+1}]}}(1 - \widetilde{\pi}_{0,j}^*)}\right)$$

Features of bound and interpretation:

1. Relates **observable values** to **unobserved structural parameter** Discussion
2. Under risk neutrality ($\phi_j = 1$): Bound becomes 0
3. Movement in risk-neutral beliefs still must correspond (on average) to agent learning about return, but now have inequality bound where $\frac{\partial \text{bound}}{\partial \phi_j} > 0$
4. Identifies min. required ϕ_j, which can be interpreted in terms of relative risk aversion if ∃ rep. agent with utility over index level:

$$\gamma_j = \frac{\phi_j - 1}{(s_{j+1} - s_j)/s_j}$$

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$$\widetilde{\mathbb{E}}_0[m_j^* - r_j^*] \leqslant {\widetilde{\pi}_{0,j}^*}^2 \left(1 - \frac{1}{\widetilde{\pi}_{0,j}^* + \underbrace{\phi_j}_{\substack{\cancel{U'(C_{\text{low}})/U'(C_{\text{high}})} \\ \mathbb{E}_t[M_T \mid R_T^m = s_j]/\mathbb{E}_t[M_T \mid R_T^m = s_{j+1}]}} (1 - \widetilde{\pi}_{0,j}^*)}\right)$$

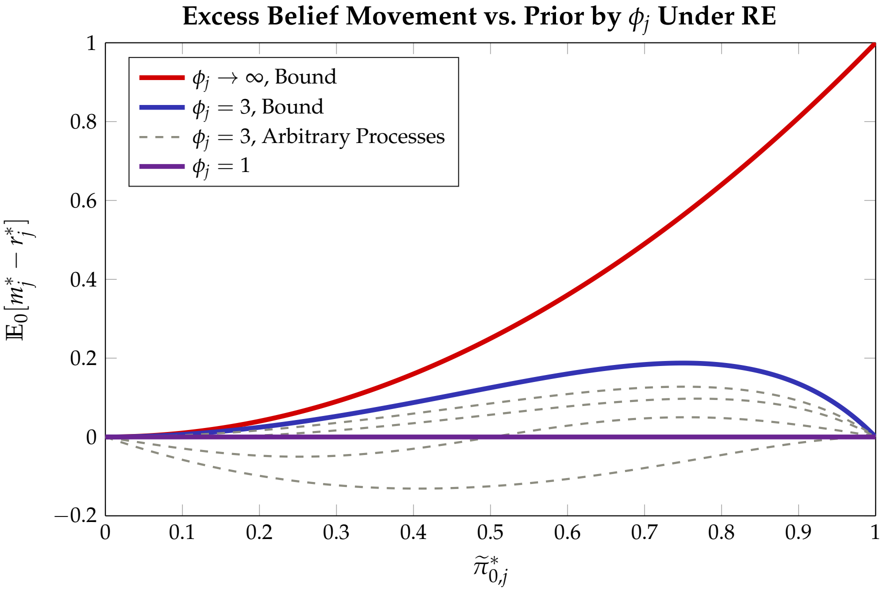
Taking $\phi_j \to \infty$:

Corollary

If $\mathbb{E}_0[m_j^* - r_j^*] > {\widetilde{\pi}_{0,j}^*}^2$, then **no** SDF process under which ϕ_j is constant can rationalize the variation in risk-neutral beliefs for the given return-state pair.

- Can have so much excess vol. that no amount of risk aversion works
- Contrast with Hansen–Jagannathan bound

Graphical Intuition: Bound in Proposition 1



Outline

1. Background & Intro

2. Theory: Restrictions Under Rational Expectations

3. Empirics: Evidence from Index Options
    - Data
    - Baseline Results
    - Channels and Robustness

4. Discussion and Conclusions

Raw Data and Risk-Neutral Beliefs

Raw data:

- Want beliefs about return on market portfolio

$\Longrightarrow$ Data on S&P 500 index option prices from OptionMetrics, 1996–2015

Details and cleaning

Measuring risk-neutral beliefs from options:

- Breeden and Litzenberger (1978): Index price V_T^m has risk-neutral CDF

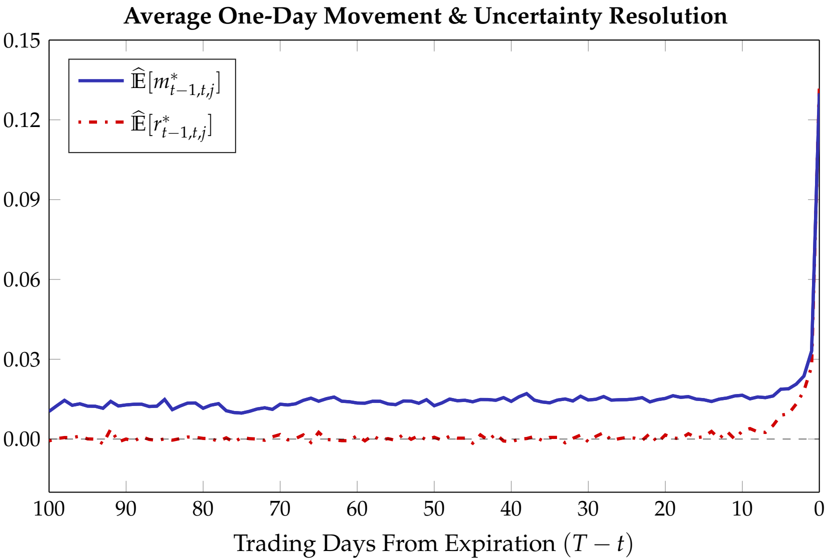
$$\mathbb{P}_t^*(V_T^m \leqslant v) = 1 + R_{t,T}^f \frac{\partial}{\partial v} \underbrace{q_t^m(v)}_{\substack{\text{option price} \\ \text{at strike } v}}$$

- Calculate $\frac{\partial}{\partial v} q_t^m(v)$ numerically following Malz (2014) Details and alternative method
- **Excess-return space**:

$$\mathcal{S}_{\text{baseline}} = \exp\{(-\infty, -11\%), -9\%, -7\%, \ldots, 7\%, 9\%, (11\%, \infty)\}$$

  - Use beliefs over $[-0.10, -0.08)$ for -9% state, ...
  - Generally use all states excluding $(-\infty, -11\%), (11\%, \infty)$

Summary: Risk-Neutral Belief Variation



Note: Empirical averages $\widehat{\mathbb{E}}[\cdot]$ calculated across all expiration dates and state pairs.

Accounting for Market Microstructure Noise

Proposition

Assume that observed $\widehat{\pi}^*_{t,j}$ is measured with error:

$$\widehat{\pi}^*_{t,j} = \widetilde{\pi}^*_{t,j} + \epsilon_{t,j},$$

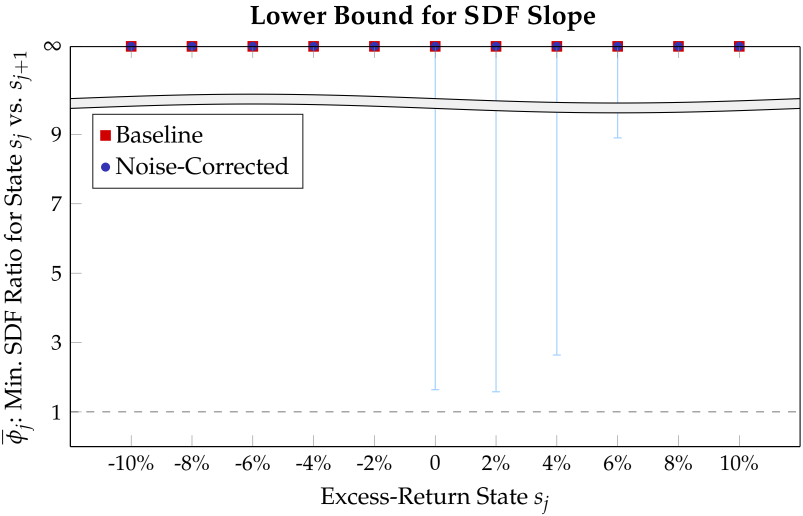
where $\widetilde{\mathbb{E}}[\epsilon_{t,j}] = 0$, $\widetilde{\mathbb{E}}[\epsilon_{t,j}\,\epsilon_{t+1,j}] = 0$, and $\widetilde{\mathbb{E}}[\epsilon_{t,j}\,\widetilde{\pi}^*_{t,j}] = 0$. Denoting observed one-period expected excess movement by $\widetilde{\mathbb{E}}_t[\widehat{m}^*_{t,t+1,j} - \widehat{r}^*_{t,t+1,j}]$, we have

$$\widetilde{\mathbb{E}}_t[\widehat{m}^*_{t,t+1,j} - \widehat{r}^*_{t,t+1,j}] = \widetilde{\mathbb{E}}_t[m^*_{t,t+1,j} - r^*_{t,t+1,j}] + 2\text{Var}(\epsilon_{t,j}).$$

To estimate $\text{Var}(\epsilon_{t,j})$, use auxiliary restriction that must hold under RE:

- Subjective forecasts must be **unbiased**: $\widetilde{\pi}_{t,j} = \mathbb{E}_t[\widetilde{\pi}_{T,j}]$
- Since $\widetilde{\pi}_{T,j} = 0$ or 1, can solve for expected forecast-error variance
- **Steps:** Conjecture $\phi_j \longrightarrow$ translate from $\widetilde{\pi}^*_{t,j}$ to $\widetilde{\pi}_{t,j} \longrightarrow$ calculate expected vs. realized forecast-error variance $\longrightarrow$ assume gap is measurement-error variance $\longrightarrow$ estimate bound $\longrightarrow$ continue until conjectured $\phi_j =$ estimated ϕ_j

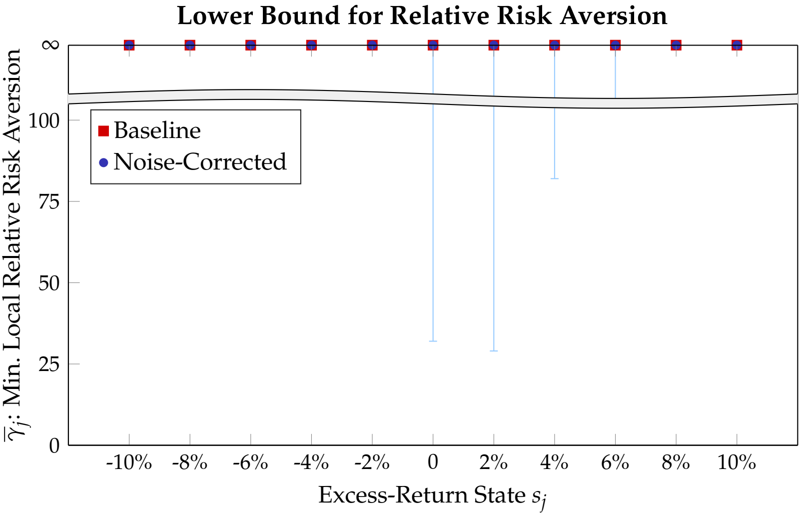
Empirical Implementation of Theoretical Bound



Note: One-sided 95% CIs use block bootstrap with bandwidth of 45 days, 5,000 draws. Details

- Aggregate across interior states: $\widehat{\phi} = \infty$, CI $[3.0, \infty)$ Formal justification

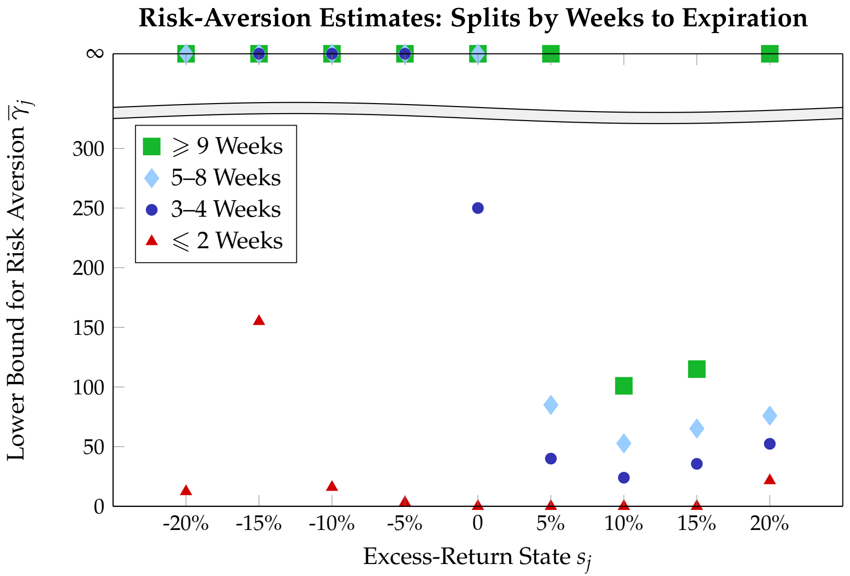
Estimation Results: Risk Aversion



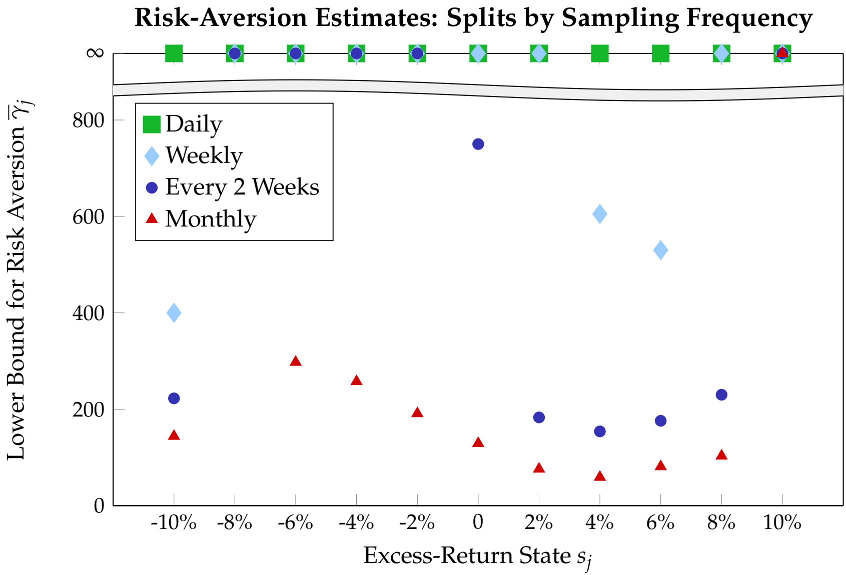
Note: One-sided 95% CIs use block bootstrap with bandwidth of 45 days, 5,000 draws. Details

- Aggregate across interior states: $\widehat{\gamma} = \infty$, CI $[102, \infty)$ Formal justification

Decompositions: Time to Expiration



Decompositions: Effects of Time Aggregation



- Aggregate monthly: $\widehat{\gamma} = 123$, CI $[97, \infty)$

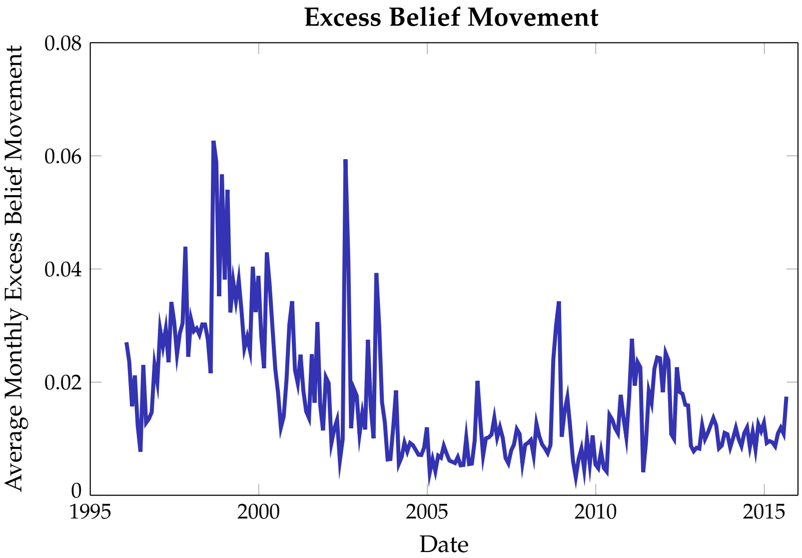
Channels: Reduced-Form Evidence

Regressions for Average Excess Belief Movement by Quarter

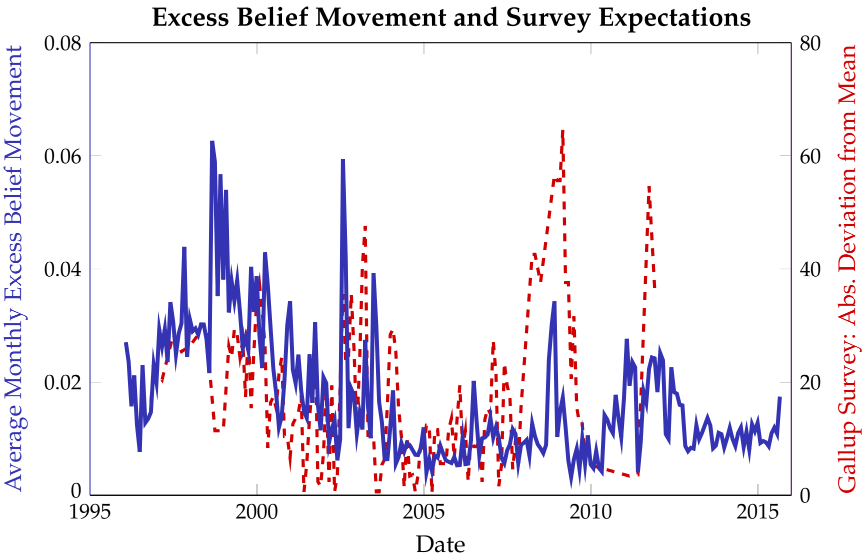
	(1)	(2)	(3)	(4)
Liquidity and Limits to Arbitrage				
Bid-Ask Spread	0.2	-0.2	-0.3*	-0.1
	(1.3)	(-0.7)	(-0.4)	(-1.0)
Broker-Dealer Leverage	-0.1	0.1	-0.0	-0.1
	(-0.4)	(0.8)	(-0.7)	(-1.7)
Volatility and Uncertainty				
VIX		0.8**	0.9***	0.6*
		(2.3)	(3.4)	(2.1)
Baker-Bloom-Davis Uncertainty		-0.3	0.1	0.2*
		(-1.2)	(1.5)	(2.2)
Returns and Valuation				
12-Month S&P Return			0.3**	0.3**
			(2.8)	(2.6)
Price to 10-Year Earnings Ratio			0.6***	0.5***
			(4.1)	(4.1)
Time Trend				-0.0*
				(-2.2)
R^2	0.07	0.34	0.72	0.73

Notes: *** $p < 0.01$; ** $p < 0.05$; * $p < 0.1$. Heteroskedasticity- and autocorrelation-robust t-statistics in parentheses, using equal-weighted periodogram estimator [Lazarus, Lewis, Stock (2017)]. All variables normalized to unit s.d. except for recession dummy and time trend. $N = 79$.

Channel: Extrapolation $\Longrightarrow$ Excess Volatility?



Channel: Extrapolation $\implies$ Excess Volatility?



Note: Gallup data (percent "bullish" less "bearish") from Greenwood and Shleifer (2014). Correlation between two series in chart: 0.30.

Robustness: Systematic Mean-Reversion vs. Noise

How real is what we're finding?

- Consider a simple statistical model for risk-neutral beliefs:

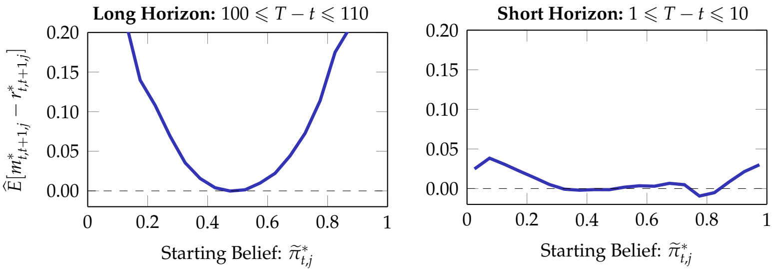
$$\widetilde{\pi}^*_{t+1,j} = \mu + \rho(\widetilde{\pi}^*_{t,j} - \mu) + \nu_{t+1}$$

- Setting $\mu = 1/2$, this model yields a prediction that:

$$\mathbb{E}[m^*_{t,t+1,j} - r^*_{t,t+1,j}] = 2(1-\rho)(\widetilde{\pi}^*_{t,j} - 1/2)^2$$

$\Longrightarrow$ should see parabola for excess movement vs. prior

Average Daily Excess Risk-Neutral Belief Movement by Starting Belief



Robustness: Conditional Transition Independence

- What if more than 1 state variable determines realization of the SDF (or MU)?
  - Then shocks to variables unspanned by asset return could violate CTI $\Longrightarrow \phi_{t,j}$ time-varying within contract

Proposition

Recall $\phi_{t,j} \equiv \mathbb{E}_t[M_T/M_t \mid R_T^m = s_j]/\mathbb{E}_t[M_T/M_t \mid R_T^m = s_{j+1}]$.

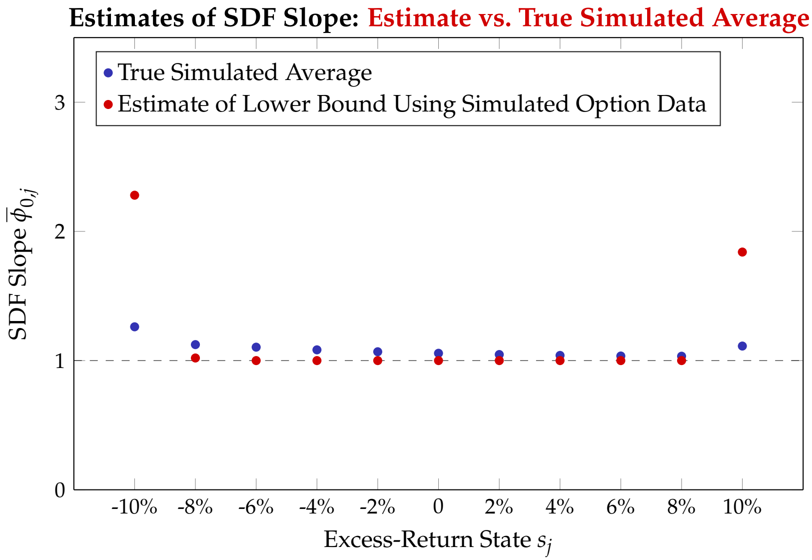
(a) If $\phi_{t,j}$ is a martingale, then the bound in Proposition 1 still applies.

(b) The bound in Proposition 1 applies to an arbitrarily close approximation within a neighborhood of $\phi_{t,j}$ being a martingale: $\forall\, \epsilon > 0, \exists\, \delta > 0$ s.t. if $|\widetilde{\mathbb{E}}_t[\phi_{t+1,j}] - \phi_{t,j}| < \delta$ a.s., then:

$$\left| \widetilde{\mathbb{E}}_0[m_j^* - r_j^*] - \widetilde{\pi}_{0,j}^{*\,2} \left(1 - \frac{1}{\widetilde{\pi}_{0,j}^* + \phi_{0,j}(1 - \widetilde{\pi}_{0,j}^*)} \right) \right| < \epsilon.$$

- How good an approximation? For now, simulations: habit formation as in Campbell & Cochrane (1999)
- Not enough variation in $\phi_{t,j}$ for simulations to match data, and bound still holds

Robustness: Habit-Formation Simulation Results



Notes: 25,000 simulations of 90-day contracts; s_0 drawn from unconditional distribution.

Outline

1. Background & Intro
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Final Notes

Summary:

- Examine restrictions on risk-neutral beliefs under RE
- Derive upper bounds for excess movement in risk-neutral beliefs, corresponding lower bounds on util. curvature needed to rationalize beliefs
- Associated empirical tests: very high curvature required to rationalize beliefs $\Longrightarrow$ overreaction to information

Possible positive models?

- Extrapolation and/or underweighting of prior relative to news?
  - Arrow (1982): *"[Evidence from Kahneman and Tversky] typifies very precisely the excessive reaction to current information which seems to characterize all the securities and futures markets."*
- Heterogeneous beliefs?

Other: Evidence from other asset classes (FI, FX), applications to real outcomes, simulations of additional models, . . .

Appendix

Derivation of Lemma

Consider the conditional expectation of the first term in the movement sum:

$$\begin{aligned}
\mathbb{E}_{t_1}[m_{t_1,t_1+1}] &= \mathbb{E}_{t_1}[(\pi_{t_1+1} - \pi_{t_1})^2], \quad \text{where } \mathbb{E}_t[\cdot] \equiv \mathbb{E}[\cdot \mid \mathcal{F}_t] \\
&= \mathbb{E}_{t_1}[\pi_{t_1+1}^2] - 2\mathbb{E}_{t_1}[\pi_{t_1+1}]\pi_{t_1} + \pi_{t_1}^2 \\
&= \mathbb{E}_{t_1}[\pi_{t_1+1}^2] - 2\pi_{t_1}\pi_{t_1} + \pi_{t_1}^2 \quad \text{(martingale prop. of Bayes' rule)} \\
&= \mathbb{E}_{t_1}[\pi_{t_1+1}^2] - \pi_{t_1}^2 + \pi_{t_1} - \mathbb{E}_{t_1}[\pi_{t_1+1}] \qquad \text{(same)} \\
&= \mathbb{E}_{t_1}[(1 - \pi_{t_1})\pi_{t_1} - (1 - \pi_{t_1+1})\pi_{t_1+1}] = \mathbb{E}_{t_1}[r_{t_1,t_1+1}].
\end{aligned}$$

Repeating for all periods and using L.I.E. yields the stated result. $\square$

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Two-State Example: Non-Constant Discount Rates

What would using the underlying price [Shiller (1981)] give us?

- Consider extreme DGP: No info revealed until date T, so $\pi_0^* = \ldots = \pi_{T-1}^*$
- Price of claim to C_T is

$$\mathbb{E}_t\left[\beta^{T-t}\frac{U'(C_T)}{U'(C_t)}C_T\right]$$

- Consider deterministic consumption stream $C_0 \neq C_1 \neq \ldots$

$\Longrightarrow$ *arbitrary* price variation as C_t changes, but *no* variation in expected payoff C_T

- Paper discusses cases with time-varying risk premia

Main Results: Implementation

$$\mathbb{E}_{0_i}[m^*_{T_i,j} - r^*_{T_i,j}] \leqslant \underbrace{\widetilde{\pi}^{*\,2}_{0_i,T_i,j}\left(1 - \frac{1}{\widetilde{\pi}^*_{0_i,T_i,j} + \phi_{i,j}(1 - \widetilde{\pi}^*_{0_i,T_i,j})}\right)}_{:= mr^*_{UB,i,j}}$$

Issue:

- Only observe one draw $m^*_{T_i,j} - r^*_{T_i,j}$ per contract, rather than $\mathbb{E}_{0_i}[m^*_{T_i,j} - r^*_{T_i,j}]$
- But $\frac{\partial^2 mr^*_{UB,i,j}}{\partial \phi^2_{i,j}} < 0$, so Jensen's inequality (and L.I.E.) give that:

$$\begin{aligned}\mathbb{E}[m^*_{T_i,j} - r^*_{T_i,j}] &\leqslant \mathbb{E}\left[\widetilde{\pi}^{*\,2}_{0_i,T_i,j}\left(1 - \frac{1}{\widetilde{\pi}^*_{0_i,T_i,j} + \phi_{i,j}(1 - \widetilde{\pi}^*_{0_i,T_i,j})}\right)\right] \\ &\leqslant \mathbb{E}\left[\widetilde{\pi}^{*\,2}_{0_i,T_i,j}\left(1 - \frac{1}{\widetilde{\pi}^*_{0_i,T_i,j} + \mathbb{E}[\phi_{i,j}](1 - \widetilde{\pi}^*_{0_i,T_i,j})}\right)\right],\end{aligned}$$

  where expectation is over all expiration dates T_i
- So min. $\mathbb{E}[\phi_{i,j}]$ solving inequality is lower bound of average ratio of SDF across states $\Longrightarrow$ info on reasonableness of pricing model required under RE
- If no such $\mathbb{E}[\phi_{i,j}]$ exists, then no SDF or risk aversion can explain $\mathbb{E}[m^*_{T_i,j} - r^*_{T_i,j}]$

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Raw Data: Details and Cleaning

Details of data:

- End-of-day prices for calls and puts, Jan. 1996–Aug. 2015, all traded strikes $\Longrightarrow$ 685 expiration dates T_i; 4,949 trading dates; 7,385,062 option prices
- Also obtain underlying index price, dividend yield, and risk-free rates from OptionMetrics, and hand-collect option settlement values from CBOE

Data cleaning:

- Drop any options with: bids of 0, Black-Scholes implied vol. more than 100%, greater than 6 months to maturity [Constantinides, Jackwerth, Savov (2013)], and any trading date–expiration date combos with fewer than 3 listed prices
- Calculate end-of-day price as average of listed bid and ask prices
- Cleaning for conditional risk-neutral probabilities: to avoid noisy measurement, only use date–state pairs meeting $\pi^*_{t,T_i,j} + \pi^*_{t,T_i,j+1} \geqslant 5\%$

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Spline Details

- Calculate $\frac{\partial}{\partial v} q_{t,T_i}(v)$ numerically following Malz (2014):
  1. Transform call and put price schedules for each date–expiration date set into Black-Scholes implied volatilities
  2. Fit clamped cubic splines to interpolate implied vols between strike prices for both calls and puts
  3. Average the calculated call and put implied vols at 1,900 strike prices
  4. Invert Black-Scholes implied volatility function to transform resulting implied vols back into call prices
  5. Numerically difference the resulting smoothed call-price schedule
- We only use Black-Scholes implied vols for smoothing and then transform vols back into prices, so doesn't require Black-Scholes model to be correct
- "Clamped" cubic spline: Sets slope of implied vol schedule to be zero at boundary strike-price values, and sets all implied vols below minimum observed strike price to value at minimum price (likewise for max.)
- This guarantees monotonically decreasing and convex call price schedule, which maintains no-arbitrage restrictions
- This is an *interpolating* spline: passes through all observed data (or *knot*) points

Alternative Smoothing Method

Bliss and Panigirtzoglou (2004) spline:

- Natural spline in implied vol–delta ($\Delta = \frac{\partial q}{\partial v}$) space, weighted by vega ($\nu = \frac{\partial q}{\partial \sigma}$)
- Smoothing spline: Penalizes squared second derivative of spline, with weight $\lambda = 0.01$ relative to deviations from observed data
- Force horizontal implied vol extrapolation by adding pseudo-data three strike intervals above/below observed strikes with implied vols equal to min./max. observed values before calculating spline
- Use out-of-the-money options (puts below current forward price, calls above)

In both cases, we set any negative probability values to 0 and renormalize the distribution accordingly. We calculate Bliss and Panigirtzoglou (2004) values to check robustness of main results.

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Estimation and Inference

Estimation:

- Define pricing-kernel ratio as before: $\phi_{i,j} \equiv \frac{\mathbb{E}_t[M_{T_i}/M_t \mid R^m_{T_i}=s_j]}{\mathbb{E}_t[M_{T_i}/M_t \mid R^m_{T_i}=s_{j+1}]} \geqslant 1$
- Main result: $\mathbb{E}[m^*_{T_i,j} - r^*_{T_i,j}] \leqslant \mathbb{E}\left[{\widetilde{\pi}^*_{0_i,T_i,j}}^2 \left(1 - \frac{1}{\widetilde{\pi}^*_{0_i,T_i,j} + \mathbb{E}[\phi_{i,j}](1-\widetilde{\pi}^*_{0_i,T_i,j})}\right)\right]$
- Denote $\overline{\phi}_j \equiv \min\left\{\mathbb{E}[\phi_{i,j}]\right\}$ solving inequality
- Sample estimator for each state pair solves moment condition:

$$\frac{1}{N}\sum_{i=1}^{N}\left[m^*_{T_i,j} - r^*_{T_i,j} - {\widetilde{\pi}^*_{0_i,T_i,j}}^2\left(1 - \frac{1}{\widetilde{\pi}^*_{0_i,T_i,j} + \widehat{\phi}_j(1 - \widetilde{\pi}^*_{0_i,T_i,j})}\right)\right] = 0$$

Inference:

- CIs using block bootstrap: Partition sample into blocks of length B days, then resample (with replacement) all paths in entire block & re-estimate
- Addresses possible temporal dependence and correlation across state pairs

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Interpretation: Priors or Updating?

- Recall: Testing null of *both* correct prior and correct updating
- Are we really finding excess volatility of beliefs, or just miscalibrated priors?

Proposition

Assuming CTI holds for the return-state pair (s_j, s_{j+1}), the effects of an incorrect physical prior, $\widetilde{\pi}_{0,j} \neq \mathbb{P}_0(R_T^m = s_j \mid R_T^m \in \{s_j, s_{j+1}\})$, are limited as follows:

(i) If $\widetilde{\pi}^*_{0,j} < \mathbb{P}_0(R_T^m = s_j \mid R_T^m \in \{s_j, s_{j+1}\})$, then under Bayesian updating, it must be the case that $\widetilde{\mathbb{E}}_0[m_j^* - r_j^*] \leqslant \max\{{\widetilde{\pi}^*_{0,j}}^2, (1 - \widetilde{\pi}^*_{0,j})^2\}$.

(ii) Otherwise, an incorrect prior cannot by itself lead to $\widetilde{\mathbb{E}}_0[m_j^* - r_j^*] > {\widetilde{\pi}^*_{0,j}}^2$.

- Thus $\widehat{\phi}_j = \infty \implies$ likely excess volatility, not just bad priors